35.1.

$$f(x) = \begin{cases} a \cdot x & \text{for } x \in [-\pi, 0] \\ b \cdot x & \text{for } x \in (0, \pi) \end{cases}, f(x + 2k\pi) = f(x), k \in \mathbb{Z}$$

I will define here

$$f_{a,b}(x) := \begin{cases} a \cdot x & \text{for } x \in [-\pi, 0] \\ b \cdot x & \text{for } x \in (0, \pi) \end{cases}, f(x + 2k\pi) = f(x), k \in \mathbb{Z}$$

to write cases of f that have a specific a and b.

The general Fourier series looks somewhat like this:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

First we should determine a_n:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx) \,\mathrm{d}x \\ &= \frac{1}{\pi} \left(\int_{-\pi}^{0} ax \cdot \cos(nx) \,\mathrm{d}x + \int_{0}^{\pi} bx \cdot \cos(nx) \,\mathrm{d}x \right) \\ &= \frac{1}{\pi} \left(\frac{a}{n^2}(1 - (-1)^n) + \frac{b}{n^2}((-1)^n - 1) \right) \\ &= \frac{b-a}{n^2\pi} \cdot ((-1)^n - 1) \end{aligned}$$

a_0 requires some special treatment, though:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \left(\int_{-\pi}^{0} ax \cdot \cos(0) \,\mathrm{d}x + \int_{0}^{\pi} bx \cdot \cos(0) \,\mathrm{d}x \right) \\ &= \frac{1}{\pi} \left(\frac{a}{2} x^2 \Big|_{-\pi}^{0} + \frac{b}{2} x^2 \Big|_{0}^{\pi} \right) \\ &= \frac{1}{\pi} \left(\frac{a}{2}(-\pi^2) + \frac{b}{2}(\pi^2) \right) \\ &= \frac{\pi}{2}(b - a) \end{aligned}$$

This leaves us the b_n:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cdot\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(\int\limits_{-\pi}^{0} ax\cdot\sin(nx)\,\mathrm{d}x + \int\limits_{0}^{\pi} bx\cdot\sin(nx)\,\mathrm{d}x\right) \\
&= -\frac{1}{n}(a\cdot(-1)^n + b\cdot(-1)^n) \\
&= -\frac{a+b}{n}\cdot(-1)^n
\end{aligned}$$

We now get the following Fourier series:

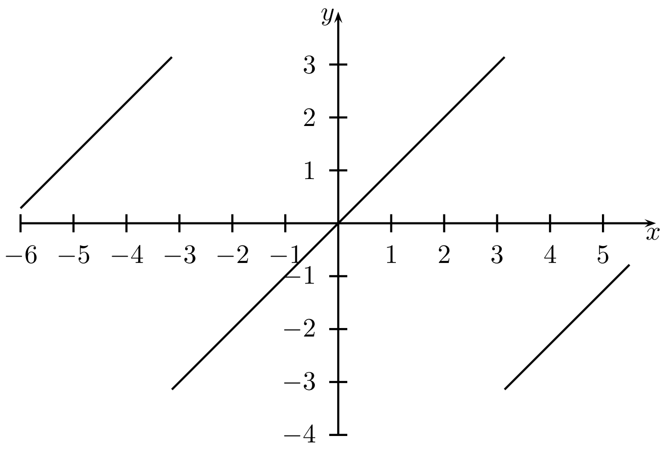
$$\begin{aligned}
s(x) &= \frac{\pi}{4}(b-a) + \sum_{n=1}^{\infty}\left(\frac{b-a}{n^2\pi}\cdot((-1)^n-1)\cdot\cos(nx) - \frac{a+b}{n}\cdot(-1)^n\cdot\sin(nx)\right) \\
&= \begin{cases} f(x) & \text{where } f(x) \text{ is continuous} \\ \frac{\pi}{2}(b-a) & \text{for } x\in\{\,k\pi \mid k\in\mathbb{Z}\} \end{cases}
\end{aligned}$$

(a)

$a = b = 1$ yields the following series:

$$\begin{aligned}
s_{1,1}(x) &= \sum_{n=1}^{\infty}\left(-\frac{2}{n}\cdot(-1)^n\cdot\sin(nx)\right) \\
&= \begin{cases} f_{1,1}(x) & \text{where } f_{1,1}(x) \text{ is continuous} \\ 0 & \text{for } x\in\{\,k\pi \mid k\in\mathbb{Z}\} \end{cases}
\end{aligned}$$

as well as the following graph:

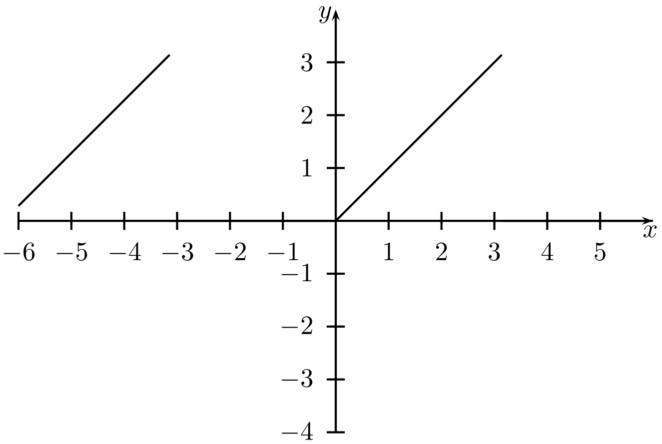


(b)

$a = 0, b = 1$ yields the following series:

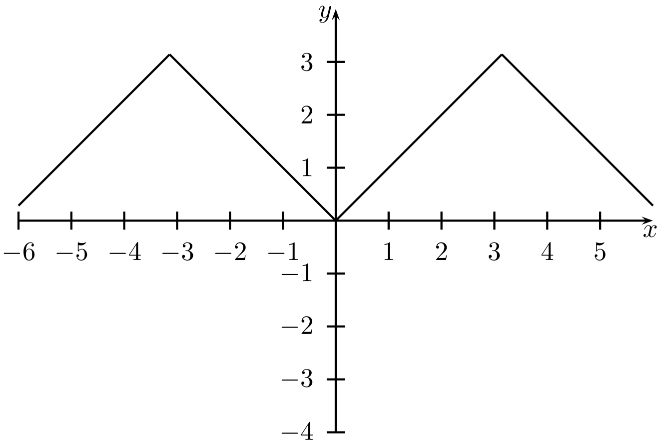
$$\begin{aligned} s_{0,1}(x) &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2\pi} \cdot ((-1)^n - 1) \cdot \cos(nx) - \frac{1}{n} \cdot (-1)^n \cdot \sin(nx) \right) \\ &= \begin{cases} f_{0,1}(x) & \text{where } f_{0,1}(x) \text{ is continuous} \\ \frac{\pi}{2} & \text{for } x \in \{ k\pi \mid k \in \mathbb{Z} \} \end{cases} \end{aligned}$$

as well as the following graph:



(c)

$a = -1, b = 1$ yields the following graph:



and the series:

$$\begin{aligned} s_{-1,1}(x) &= \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi}\cdot\left((-1)^n - 1\right)\cdot\cos(nx)\right) \\ &= \begin{cases} f_{-1,1}(x) & \text{where } f_{-1,1}(x) \text{ is continuous} \\ & \text{for } x \in \{\, k\pi \mid k \in \mathbb{Z}\} \end{cases} \end{aligned}$$

This leaves us for

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

which is essentially the Fourier series determined above for $x = 0$.

35.2

Let $l > 0$ and

$$f(x) = x^2, x \in [-l, l]$$

$$f(x + 2lk) = f(x), k \in \mathbb{Z}$$

The general Fourier series for this case is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

with

$$\begin{aligned} a_0 &= \frac{1}{l}\int_{-l}^{l} x^2 \,\mathrm{d}x \\ &= \frac{1}{3l}\,x^3\Big|_{-l}^{l} \\ &= \frac{2l^2}{3} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l} x^2\cos\frac{n\pi x}{l}\,\mathrm{d}x \\ &= \frac{4l^2\cos(\pi n)}{\pi^2 n^2} + \frac{2l^2\left(\pi^2 n^2 - 2\right)\cdot\sin(\pi n)}{\pi^3 n^3} \\ &= \frac{4l^2\cdot(-1)^n}{\pi^2 n^2} \end{aligned}$$

b_n is simply zero, because $f(x)$ is an even function and hencefoth can only be represented by a sum of other even functions, so every coefficient before $\sin\frac{n\pi x}{l}$ has to be 0.

This leads us to the final result:

$$f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2 \cdot (-1)^n}{\pi^2 n^2} \cdot \cos\frac{n\pi x}{l}$$

35.3

(b)

$$V = C[-1,1]$$

$$\varphi(f,g) := \int\limits_{-1}^{1} f(x) \cdot g(x) \,\mathrm{d}x$$

$$\mathfrak{a}_1 = f_1(x) = 1, \mathfrak{a}_2 = f_2(x) = x, \mathfrak{a}_3 = f_3(x) = x^2$$

$$\mathfrak{e}_1 = \frac{\mathfrak{a}_1}{\|\mathfrak{a}_1\|_\varphi} = \frac{1}{\sqrt{\int\limits_{-1}^{1} 1 \,\mathrm{d}x}} = \frac{1}{\sqrt{2}}$$

$$\begin{aligned}
\mathfrak{b}_2 &= \mathfrak{a}_2 - \varphi(\mathfrak{a}_2, \mathfrak{e}_1) \cdot \mathfrak{e}_1 \\
&= x - \int\limits_{-1}^{1} x \cdot \frac{1}{\sqrt{2}} \,\mathrm{d}x \cdot \frac{1}{\sqrt{2}} \\
&= x - \frac{1}{4} x^2 \Big|_{-1}^{1} \\
&= x
\end{aligned}$$

$$\begin{aligned}
\mathfrak{e}_2 &= \frac{\mathfrak{b}_2}{\|\mathfrak{b}_2\|_\varphi} \\
&= \frac{x}{\|x\|_\varphi} \\
&= \frac{x}{\sqrt{\int\limits_{-1}^{1} x \,\mathrm{d}x}} \\
&= \frac{\sqrt{6}}{2} x
\end{aligned}$$

$$\begin{aligned}\mathfrak{b}_3 &= \mathfrak{a}_3 - \varphi(\mathfrak{a}_3, \mathfrak{e}_1) \cdot \mathfrak{e}_1 - \varphi(\mathfrak{a}_3, \mathfrak{e}_2) \cdot \mathfrak{e}_2 \\ &= x^2 - \int\limits_{-1}^{1} \frac{1}{\sqrt{2}} x^2 \,\mathrm{d}x \cdot \frac{1}{\sqrt{2}} - \int\limits_{-1}^{1} \frac{\sqrt{6}}{2} x^3 \,\mathrm{d}x \cdot \frac{\sqrt{6}}{2} x \\ &= x^2 - \frac{1}{3}\end{aligned}$$

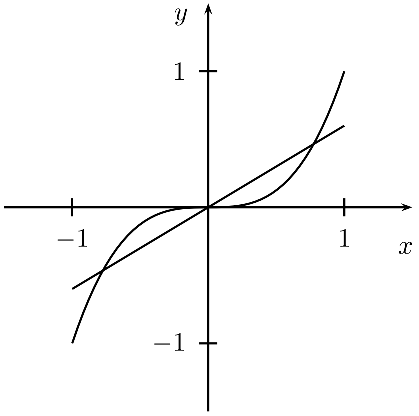
$$\begin{aligned}\mathfrak{e}_3 &= \frac{\mathfrak{b}_3}{\|\mathfrak{b}_3\|_\varphi} \\ &= \left(x^2 - \frac{1}{3}\right) \cdot \left(\int\limits_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 \mathrm{d}x\right)^{-\frac{1}{2}} \\ &= \frac{3\sqrt{10}}{4}\left(x^2 - \frac{1}{3}\right)\end{aligned}$$

35.4

$$f(x) = x^3 \in C[-1, 1]$$

$$f_1(x) = \sqrt{\frac{1}{2}}, f_2(x) = \sqrt{\frac{3}{2}} \cdot x, f_3(x) = \sqrt{\frac{5}{2}} \cdot \left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

$$\begin{aligned}\widetilde{f}(x) &:= \varphi(f, f_1) \cdot f_1(x) + \varphi(f, f_2) \cdot f_2(x) + \varphi(f, f_3) \cdot f_3(x) \\ &= \int\limits_{-1}^{1} x^3 \cdot \sqrt{\frac{1}{2}} \,\mathrm{d}x \cdot \sqrt{\frac{1}{2}} + \int\limits_{-1}^{1} x^3 \cdot \sqrt{\frac{3}{2}} \cdot x \,\mathrm{d}x \cdot \sqrt{\frac{3}{2}} \cdot x \\ &+ \int\limits_{-1}^{1} x^3 \cdot \sqrt{\frac{5}{2}} \cdot \left(\frac{3}{2}x^2 - \frac{1}{2}\right) \mathrm{d}x \cdot \sqrt{\frac{5}{2}} \cdot \left(\frac{3}{2}x^2 - \frac{1}{2}\right) \\ &= 0 + \frac{3}{5}x + 0 \\ &= \frac{3}{5}x\end{aligned}$$



$$\begin{aligned}
\left\|\widetilde{f}-f\right\|_{\varphi} &= \sqrt{\|f\|_{\varphi}^{2}-\varphi(f,f_1)^2-\varphi(f,f_2)^2-\varphi(f,f_2)^2} \\
&= \sqrt{\int\limits_{-1}^{1} x^6\,\mathrm{d}x-\left(\int\limits_{-1}^{1} x^3\cdot\sqrt{\frac{1}{2}}\,\mathrm{d}x\right)^2-\left(\int\limits_{-1}^{1} x^3\cdot\sqrt{\frac{3}{2}}\cdot x\,\mathrm{d}x\right)^2-\left(\int\limits_{-1}^{1} x^3\cdot\sqrt{\frac{5}{2}}\cdot\left(\frac{3}{2}x^2-\frac{1}{2}\right)\mathrm{d}x\right)^2} \\
&= \sqrt{\frac{2}{7}-\frac{6}{25}} \\
&= \frac{2\sqrt{14}}{35} \\
&\approx 0.2138
\end{aligned}$$

$$\left\|\widetilde{f}-f\right\|_{\infty} = \max_{x\in[-1,1]}\left|\widetilde{f}-f\right|$$

So all we have to do here is getting all extrema of $\widetilde{f}-f$:

$$\left(\frac{3}{5}x-x^3\right)' = -3x^2+\frac{3}{5}$$

So we get four x we have to take a look at:

$$\begin{aligned}
x=-\frac{\sqrt{5}}{5} &\Rightarrow -\frac{2\sqrt{5}}{25} \\
x=\frac{\sqrt{5}}{5} &\Rightarrow \frac{2\sqrt{5}}{25} \\
x=-1 &\Rightarrow \frac{2}{5} \\
x=1 &\Rightarrow -\frac{2}{5}
\end{aligned}$$

Thus we get:

$$\left\|\widetilde{f}-f\right\|_{\infty} = \max_{x\in[-1,1]}\left|\widetilde{f}-f\right| = \frac{2}{5}$$